Math 249 Lecture 26 Notes

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1 The Lagrange Inversion Formula

We want to find the compositional inverse of a formal power series. To get there, we need a fact about trees.

1.1 Another Cayley tree theorem

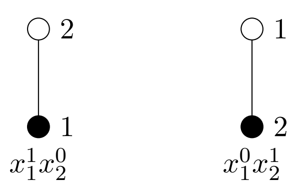
Recall from last time that we had Cayley's formula, which said that $t_n = n^{n-1}$ is the number of trees on n vertices.

Theorem 1.1 (Cayley)**.** *Given a labeled tree T on n vertices, let $c_T(i)$ be the number of children of node i in T. Then*

$$\sum_T \prod_i x_i^{c_T(i)} = (x_1 + \cdots + x_n)^{n-1}.$$

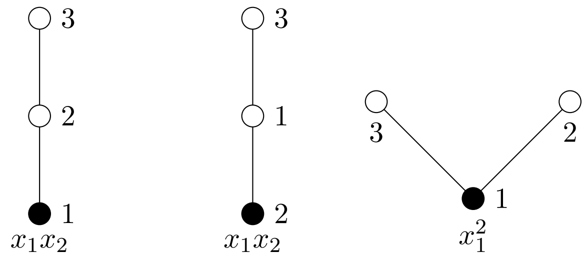
Before we provide the proof, let's look a few examples to see this in action. This really is a remarkable fact.

Example 1.1. Let $n = 2$.



We get $x_1^1 x_2^0$ and $x_1^0 x_2^1$, so $x_1 + x_2 = (x_1 + x_2)^1$.

Example 1.2. Let $n = 3$.



We get $2x_1x_2 + 2x_2x_3 + 2x_1x_3 + x_1^2 + x_2^2 + x_3^3 = (x_1 + x_2 + x_3)^2$.

Proof. We proceed by induction on n. Define the generating function

$$T_n(x_1, \dots, x_n) = \sum_T \prod_i x_i^{c_T(i)}.$$

This is a homogeneous polynomial of degree $n-1$, and so is $(x_1 + \cdots + x_n)^{n-1}$. It is also symmetric in the x_i. Since it is homogeneous of degree $n-1$, every term omits at least one variable; again, so does $(x_1 + \cdots + x_n)^{n-1}$.

As a result of these observations, it is sufficient to show that

$$T_n(x_1, \dots, x_n)|_{x_i=0} = (x_1 + \cdots + x_n)^{n-1}|_{x_i=0} \quad \forall i.$$

By symmetry, we can just do the $x_n = 0$ case. Then $T_n(x_1, \dots, x_{n-1}, 0)$ just enumerates the trees in which x_n is a leaf. What happens when you add the vertex n as a leaf? The number of children of a vertex increases by 1, and we can do this for any of the vertices $1, \dots, n-1$. So we get

$$T_n(x_1, \dots, x_{n-1}, 0) = (x_1 + \cdots + x_{n-1})T_{n-1}(x_1, \dots, x_{n-1}),$$

and applying the inductive hypothesis concludes the proof. □

Corollary 1.1. *The number of rooted trees on vertices $\{1, \dots, n\}$ with $c_T(i) = k_i$ for all i (for given k_i) is the multinomial coefficient $\binom{n-1}{k_1, k_2, \dots, k_n}$.*

Proof. This is the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in $(x_1 + \cdots + x_n)^{n-1}$. □

1.2 Lagrange inversion

Let $T(x;, a_0, a_1, \dots)$ be a mixed generating function for species of rootes trees, weighted by $\prod_{v \in S} a_{C_T(v)}$. That is,

$$
\begin{aligned}
T(x;,a_0,a_1,\dots) &= \sum_{n=1}^{\infty} \sum_{k_1+\dots+k_n=n-1} \binom{n-1}{k_1,k_2,\dots,k_n} a_{k_1}\cdots a_{k_n} \frac{x^n}{n!} \\
&= \sum_{n=1}^{\infty} \sum_{k_1+\dots+k_n=n-1} \frac{a_{k_1}}{k_1!}\cdots\frac{a_{k_n}}{k_n!}\frac{x^n}{n}.
\end{aligned}
$$

If we let $A(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + \dots = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$, then the coefficient

$$
\langle x^n \rangle T(x;a_0,a_1,\dots) = \frac{1}{n} \left\langle x^{n-1} \right\rangle A(x)^n.
$$

Think of $A(x)$ as the mixed generating function for E with structure on $|S| = k$ weighted by a_k. This is a sort of "generic species."

Recall that $T \cong X(E \circ T)$, which gives us that $T(x) = xe^{T(x)}$. Last time, we saw that this gave that $T(x) = \sum_{n=1}^{\infty} n^{n-1} \frac{x^n}{n!}$. If you weight the children of a vertex, you get that

$$
T(x;a_0,a_1,\dots) = xA(T(x;a_0,a_1,\dots)).
$$

Solving this equation makes sense for any formal power series $A(x)$. Assume that a_0^{-1} exists so $A(x)^{-1}$, the multiplicative inverse of $A(x)$, makes sense. This means that $T/A(T) = x$, which gives us that

$$
(x/A(x)) \circ T = x.
$$

In general, given $G(x)$ with $G(0) = 0$, we can say $G(x) = x/A(x)$, where $A(x) = x/G(x)$, which makes sense because $G(x)/x$ is a formal power series and has a multiplicative inverse.

Theorem 1.2 (Lagrange inversion)**.** *If $G(x) = x/A(x)$, with $G(0) = 0$ and a_0^{-1} exists, then*

$$
\langle x^n \rangle G^{\langle -1 \rangle}(x) = \frac{1}{n} \left\langle x^{n-1} \right\rangle A(x)^n.
$$

Example 1.3. $T(x) = xe^{T(x)}$, so $T(x)e^{-T(x)} = x$. Then

$$
T(x) = G(x)^{\langle -1 \rangle},
$$

where $G(x) = xe^{-x}$ and $A(x) = e^x$. So

$$
\langle x^n \rangle T(x) = \frac{1}{n} \left\langle x^{n-1} \right\rangle e^{nx} = \frac{1}{n} \frac{n^{n-1}}{(n-1)!} = \frac{1}{n!} n^{n-1}.
$$

The left hand side is $t_n/n!$, so $t_n = n^{n-1}$.

Example 1.4. Let T_p be the species of rooted plane trees.

$$T_p = X(L \circ T_p)$$

So $T_p(x) = G(x)^{\langle -1 \rangle}$, where $G(x) = x(1-x)$; this makes $A(x) = \frac{1}{1-x}$. So

$$\frac{1}{n!} t_p(n) = \langle x^n \rangle \, T_p(x) = \frac{1}{n} \left\langle x^{n-1} \right\rangle (1-x)^{-n} = \frac{1}{n} \left\langle \begin{matrix} n \\ n-1 \end{matrix} \right\rangle = \frac{1}{n} \binom{2n-2}{n-1}.$$

These are the *Catalan numbers*, shifted over by 1. So the number of unlabeled plane trees on n vertices is C_{n-1}.